Predictable or Chaotic?

Orbits vs. Weather

New Mexico
Supercomputing Challenge
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Team 48
Los Alamos High School

Team Member:
Theodore Petersen

Teacher Sponsor:
Adam Drew

Project Mentor:
Mark Petersen
TABLE OF CONTENTS

Executive Summary 3
Introduction 4
Description 4
Results 6
Conclusions 18
Acknowledgements 18
References 18
EXECUTIVE SUMMARY

This project investigates the predictability of planetary and weather systems. A computer code was written to model each system. A simple analogy of weather prediction is the Lorenz attractor, a set of differential equations that model atmospheric convection. A large number of simulations were run with many distinct initial conditions. It was demonstrated that no matter how close together the initial points are set, the paths will diverge if given enough time. The solar system model applies gravitational properties such as Newton’s second law and Newton’s law of universal gravitation. After giving all of these planets accurate initial velocities and positions, this simple model of the solar system was used to predict planetary orbits. By perturbing the initial positions of Earth and Jupiter in a large number of simulations, it was shown that planetary orbits are relatively stable. Although large perturbations in the initial locations cause large alterations of the orbit, small perturbations cause small changes.

These models show the fundamental difference between chaotic and predictable systems: weather is only predictable for a short period of time and is very sensitive to initial conditions, while planetary orbits are predictable for long durations and are not very sensitive to small perturbations.
INTRODUCTION

In 1963, Ed Lorenz derived equations with the intent to simplify some weather forecasting formulas [2]. Lorenz stumbled upon this revolutionary example of a chaotic system without having any idea of how his model would play such a big part within the scientific and mathematical community. Before Lorenz had made this discovery, the scientific community had thought that all systems were predictable if sufficient information was known. These equations, widely known as the Lorenz attractor, led to the discovery of chaos theory. The attractor demonstrated the ultimate importance that sensitivity to initial conditions played within chaotic systems. Within the three dimensional system of the Lorenz attractor, even if the initial spread between the points is miniscule, the points will become separated in an unpredictable manner given enough time [1].

In order to have a comparison to the Lorenz attractor, a solar system model was created to demonstrate the differences between the two systems. The planets have been orbiting the sun for billions of years, so the solar system appears to be a very stable system. A mathematical model of the planets can be created by using fundamental laws of gravitational pull, Newton's second law, and acceleration. With the initial locations, masses, and velocities of the planets, the model computes the planets’ trajectories. Within this basic system, each body acts and reacts to the gravitational pull of all the other objects, just like the actual solar system. Because the solar system is a predictable system, if one planet is slightly perturbed, then the other planets within the system should be disturbed very little, in comparison to a chaotic system, where similar paths would ultimately diverge.

DESCRIPTION

Solar System Model

Newton's law of universal gravitation [6] is

\[ F = G \frac{m_1 m_2}{d^2} \]

where \( F \) is force, \( G \) is the gravitational constant, \( m \) is the mass of an object, and \( d \) is the distance between objects,

\[ d = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2} \]

Newton’s second law is

\[ F = ma \]

where \( a \) is the acceleration. If there are multiple forces applied to an object, Newton’s second law is
\[
\sum_{i=1}^{N} F_i = ma
\]

where \( N \) is the number of forces that are acting on an object. Substituting in Newton’s law of universal gravitation and solving for acceleration of a single body,

\[
a_j = G \sum_{i=1}^{N} \frac{m_i}{d^2}
\]

A tangent line approximation was used in the python computer code [4] to compute the derivatives.

Acceleration is computed as

\[
a = \frac{dv}{dt} \approx \frac{v_{n+1} - v_n}{t_{n+1} - t_n}
\]

and velocity is

\[
v = \frac{dx}{dt} \approx \frac{x_{n+1} - x_n}{t_{n+1} - t_n}
\]

where \( x \) is the location and \( n \) is the time index. In three dimensions, the code computes the acceleration of each planet in reaction to all the other bodies within the system. The velocity for these bodies is based on the acceleration and at every time step, the location is based on the velocity.

At each time step in the python code computes the following:

\begin{enumerate}
  \item loop in time:
  \item loop over all planets:
    \begin{enumerate}
      \item compute acceleration based on forces of other planets
        \[
a_n = G \sum_{i=1}^{N} \frac{m_i}{d^2}
        \]
      \item compute velocity from acceleration
        \[
v_{n+1} = v_n + a_n(t_{n+1} - t_n)
        \]
      \item compute new location from velocity
        \[
x_{n+1} = x_n + v_{n+1}(t_{n+1} - t_n)
        \]
    \end{enumerate}
\end{enumerate}

Lorenz Model

According to Wolfram MathWorld, “The Lorenz attractor is an attractor that arises in a simplified system of equations describing the two-dimensional flow of fluid of uniform depth, with an imposed temperature difference, under gravity, with buoyancy, thermal diffusivity, and kinematic viscosity” [5]. The Lorenz attractor is the system of differential equations
where $x$ is the intensity of the convection current, $y$ is the temperature of the currents going up and the currents going down, $z$ is the difference in vertical temperature profile from linearity. The constants, $\sigma, \rho, \beta$, are dimensionless parameters related to the strength of heating, the viscosity, the diffusivity, and the geometry of the domain.

A computer model of the Lorenz equations was created using Python code. Using the tangent line approximation to step forward in time,

$$\frac{dx}{dt} \approx \frac{x_{n+1}-x_n}{t_{n+1}-t_n}$$

When solving for $x$ at the new time,

$$x_{n+1} = x_n + \frac{dx}{dt}(t_{n+1} - t_n)$$

it is possible to solve for the new position of $x$ at the next time step. The variables $y$ and $z$ are stepped forward in a similar manner.

The initial positions of $x, y$ and $z$ are controlled within the code as follows. The points are randomly placed within a conceptual cube of space. The size and the locations for this cube are set within the code.

At each timestep in the code, a spread is calculated as follows. The center of the cluster of points is computed by taking the average location of all the points. The spread is the average distance of all the points to the center.

**RESULTS**

**Solar System Model**

The initial positions and velocities of the planets were acquired from the JPL Horizons website [3]. Using these, it was possible to create a realistic simulation of the solar system. Figure 1 is a three-dimensional image of this simulation. From the start date of 1/4/2016, a simulation was run for one Earth year with a time step of one Earth day (fig. 2). It was confirmed that Earth returned to the same location after one year, showing that the model was working correctly. The simulation was run out for a duration of twenty years in order to show the orbits of the outer planets (fig. 3).

In order to see the effects of perturbing the solar system, three sets of experiments were conducted where the Sun, the Earth, and Jupiter were perturbed in each. When the Sun was
perturbed by 0.5 AU, (astronomical units) the planets had very large orbital adjustments, but continued to orbit around the Sun (fig. 8). In the next experiments, the initial locations of Earth and Jupiter were perturbed randomly within a set cube of space. This cube's dimensions varied from 1 AU to $1 \times 10^{-6}$ AU in increments of $1 \times 10^{-1}$. Within these experiments, a group of 10 simulations were run at each separation distance. Figure 4 and 5 show the average spread of Earth amongst these 10 simulations when Earth is perturbed, (fig. 5) and when Jupiter is perturbed (fig. 4). These results show that large initial perturbations lead to large adjustments in Earth’s orbit, and that small perturbations lead to small adjustments. This demonstrates that the solar system is predictable on a time scale of at least 300 years.

**Lorenz Model**

Using the Lorenz equations, figures 9-11 show three different moments in time when the particle paths are plotted in three dimensional parameter space. Much like the perturbation of the planets, the initial points are randomly placed within a conceptual cube of x-y-z space, and then run out to the desired length of time.

Figures 12-14 have 6 different groups of 10 simulations within each graph. Each line on the graph has a different size cube of initial locations, ranging from 10 to $1 \times 10^{-6}$ in increments of ten. Other than figure 13a, these are displayed in the form of a log plot so that the initial spread of each simulation is more visible. The y axis on these plots show the average separation of particles. From these plots, it is easy to see that even if the initial points are very close together, given enough time, they will separate in an unpredictable manner.
Figure 2b

Date: 09/29/2016

Figure 3

Date: 12/01/2035
Figure 4: Each line is average of 10 sims. where Jupiter is perturbed.

Figure 5: Each line is average of 10 sims. where Earth is perturbed.
Figure 12a: Each line is average separation of 10 particles - 6 sims.

Figure 12b: log plot of Figure 12a
Figure 13a

Figure 13b
CONCLUSIONS

In this project, two systems of differential equations were tested for predictability. It was found that the Lorenz attractor was a chaotic system. No matter how close the initial points were placed together they separate in an unpredictable manner. On the other hand, the planetary orbits were stable to small perturbations. This draws the conclusion that the solar system is predictable.

In closing, the solar system is a predictable system. This means that it is stable even with small anomalies in the system. For example, it is possible to predict an eclipse hundreds of years from now, down to the minute that it starts, even if there is an unexpected meteor that enters the solar system. The Lorenz attractor is a chaotic system. Because of this, the system can only be predicted for a limited amount of time. After an extended amount of time, the system becomes too chaotic to predict. For example, weather is only predictable for about ten days, and even then it is not entirely accurate.

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REFERENCES