Virtually Reconstructing an Ancient Musical Instrument

New Mexico

Supercomputing Challenge

Final Report

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Welch Homeschool

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Executive Summary

Archaeomusicology is a relatively new field of study aimed at reconstructing musical traditions to improve our knowledge of ancient cultures [1]. A significant part of this endeavor is researching how an instrument's dimensions affect the frequencies it produces. The chelys, an ancient Greek lyre made from a tortoise shell, has previously been studied by construction of physical models. However, in building on these studies, this project takes a different approach. Instead of approximating the shape of the sound box and building a physical model—a process both inefficient and expensive [2]—I derive a system of equations describing the instrument's sound box and implement it to generate a mesh for performing finite element analysis. Unlike previous methods [3][4], this program allows researchers to modify the chelys's shape and size digitally and, thus, more easily study how changing its dimensions impacts the sound it would have produced.

Having built this program, I am now using it to study the following:

1) how frequency output is affected by the sound box's dimensions;

2) why instrument makers may have switched to carving lyres from wood rather than simply crafting them with tortoise shells;

3) what dimensions would make for the best-quality chelys to be played today.

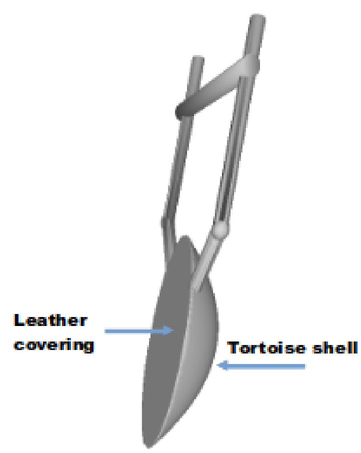
Figure 1: Image of reconstructed chelys [5].

Introduction

Finite element analysis (FEA) is a method of analyzing how a material deforms when a force (in this case, a sound wave) is exerted on it [6]. By modeling the material as a system of springs with a certain stiffness (Young's modulus) and directional response to strain (Poisson's ratio), FEA can calculate the frequencies at which the material will vibrate in response to a force. These frequencies, known as eigenfrequencies, are the pitches at which an instrument of a given material and size will naturally resonate. If an instrument is tuned such that it produces a pitch other than one of its eigenfrequencies, the sound will die away much faster, and the instrument will be of a lower quality. On the other hand, if the eigenfrequencies are close to the pitches each string is tuned to, the sound output will be louder and resonate for longer [7].

This project primarily focuses on deriving a geometrical model and performing finite element analysis on the sound box of the oldest Greek lyre, the chelys. As a 2500-year-old instrument, there is little archaeological evidence of the chelys remaining; thus, its exact size and shape are open to investigation [3]. Our knowledge of the instrument mostly comes from depictions on pottery (Figure 2a) and textual evidence, such as the 4th Homeric Hymn describing its mythical invention by the Greek god Hermes [8].

(a) Pottery depicting chelys player. [9]



(b) Diagram of chelys (author's work).

Figure 2: The Greek chelys.

The chelys is unusual in that its sound box, as shown in Figure 2b, was made of a tortoise shell with a leather covering over the opening [3]. Therefore, the chelys could have been any of a relatively wide range of sizes; however, the Greeks may have picked only certain dimensions of the tortoise shell to optimize for the best-quality instrument. Alternatively, players may have eventually replaced tortoise shells with wood so as to better control its shape and the sound it produced. While wood can be carved to a very particular shape, physically changing a tortoise shell's dimensions is much harder and inexpedient. My tool can be used to study the effects of changing both size and material on the resultant eigenfrequencies without having to physically change the dimensions.

Methodology

To derive the geometrical model of this sound box, the flat, leather side of the instrument is approximated as an ellipse (Figure 3; Equation 1), and the curved surface is approximated as a set of parabolas, where one runs parallel to the major axis of the ellipse (Figure 4; Equations 2-3), and the others run perpendicular to the major axis (Figure 5; Equations 4-6). To derive the equations, several criteria were used:

1) Each parabola must intersect with the ellipse.

2) The vertices of the parabolas perpendicular to the major axis of the ellipse must intersect with the parabola parallel to the major axis.

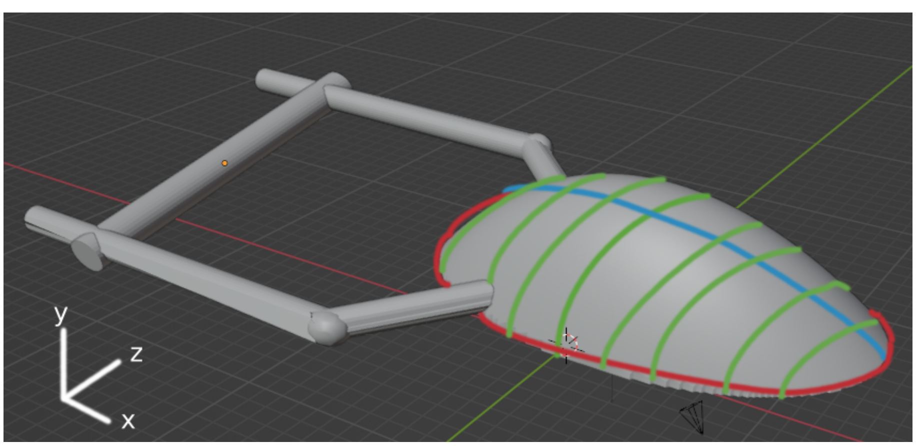
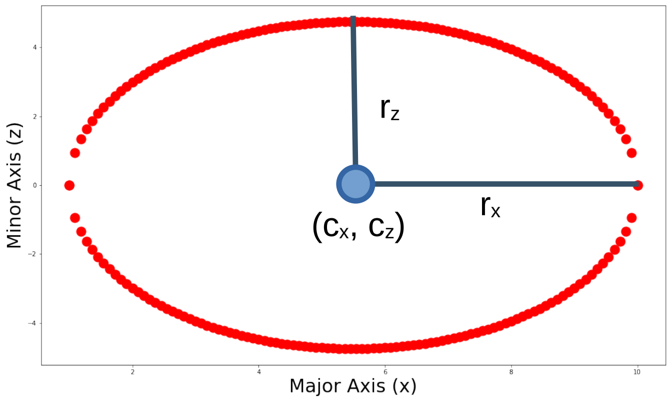


Figure 3: Diagram showing geometrical model.

Below are the derived equations with their respective graphs.

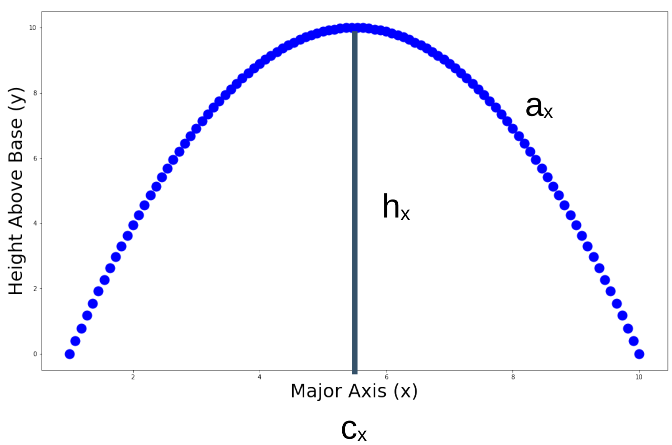
Equation 1 is the standard formula of an ellipse, set in the x-z plane, where c_x, r_x, c_z, and r_z are constants describing the shape of the ellipse.



$$1 = \left(\frac{x - c_x}{r_x}\right)^2 + \left(\frac{z - c_z}{r_z}\right)^2 \tag{1}$$

Figure 4: Elliptical base of tortoise shell.

The parabola parallel to the major axis is described by the standard formula of the parabola, where a_x, which relates to the curvature, must also be derived for the parabola and ellipse to intersect.

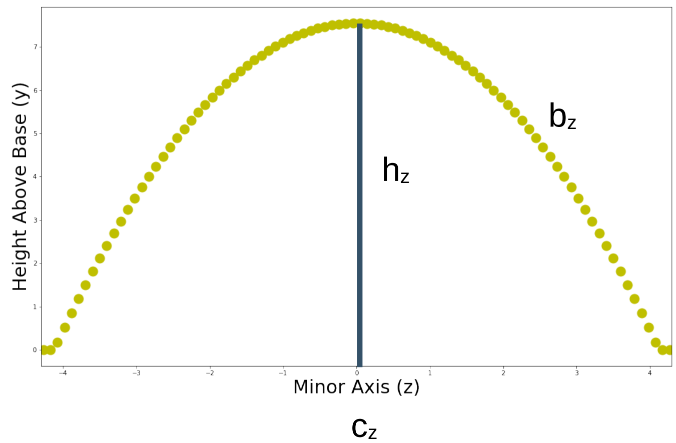


$$y(x) = a_x(x - c_x)^2 + h_x \tag{2}$$

$$a_x = -\frac{h_x}{r_x^2} \tag{3}$$

Figure 5: Parabola parallel to major axis.

Finally, the shape of each parabola perpendicular to the major axis (Equation 4) depends on where it is relative to the x-axis. As such, equations for the curvature parameter (b_z) and height of the parabola (h_z) must also be derived.



$$y(x) = b_z(z - c_z)^2 + h_z \tag{4}$$

$$h_z = \left(-\frac{h_x}{r_x^2}\right)(x - c_x)^2 + h_x \tag{5}$$

$$b_z = -\frac{h_z}{r_z\left(1 - \left(\frac{x - c_x}{r_x}\right)^2\right)} \tag{6}$$

Figure 6: Parabola perpendicular to major axis.

The next step is to use the equations in a computational model. This computational model can be viewed as a tool that inputs the sound box's desired length, width, and height and outputs a mesh that can be used in the frequency analysis. To accomplish this, a set of coordinates within the boundaries of the sound box are defined using the derived equations and then meshed using the Python library *Gmsh* [10]. A step-by-step process is given below: 1) Numpy functions were used to pick points evenly spaced along the major axis;

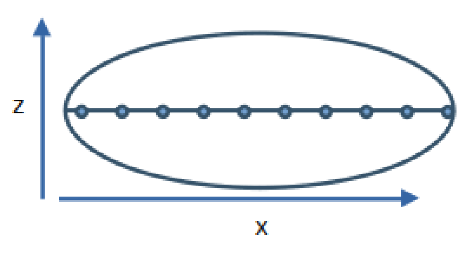


Figure 7: Evenly-spaced points defined along major axis.

2) For each point on the major axis, a column of points was defined perpendicular to the major axis;

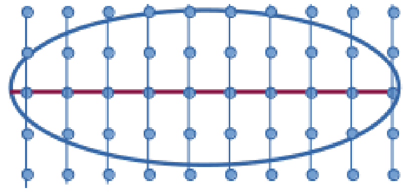
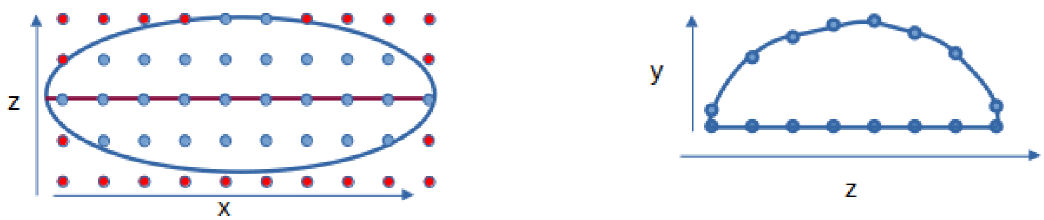


Figure 8: Points defined in columns parallel to major axis.

3) If a given point was within the bounds of the ellipse (determined by Equation 1), Equation 2 was solved to define where on the surface of the sound box that point would be;



(a) Determining which points are within the bounds of the ellipse.

(b) Finding y-coordinate for each point.

Figure 9: Step 3.

4) Using *Gmsh*, the coordinates were connected to form triangular surfaces.
The fact that the mesh is constructed correctly, as seen in Figure 10, validates my geometrical model.

Once this tetrahedral mesh was created, it could then be inputted into the second tool in the program, which performs FEA and outputs the given sound box's eigenfrequencies. This second tool is a software package known as *Calculix*; however, several steps had to be taken before using it. *Calculix* takes material properties as well as a mesh as its inputs [11]. However, since the sound box has two materials, tortoise shell and leather, the program also required defining which parts of the mesh had which material properties. In addition, the boundary conditions, the points where the sound box is held fixed and cannot

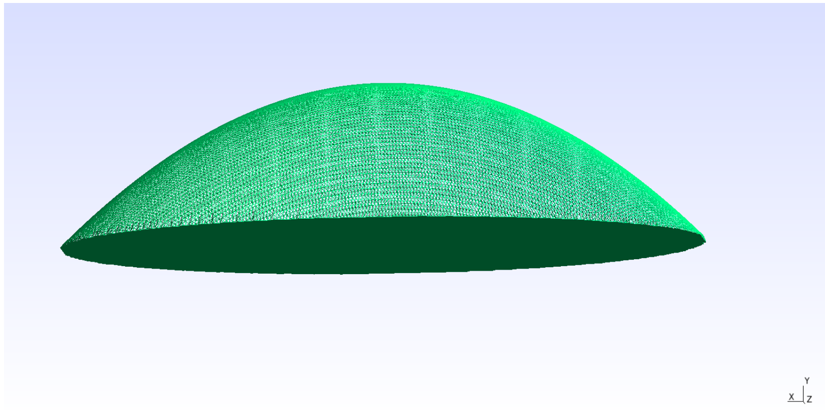


Figure 10: Example mesh.

vibrate, had to be defined. Material properties inputted into *Calculix* are given in Figure 11 [12][13][14][15][16][17].

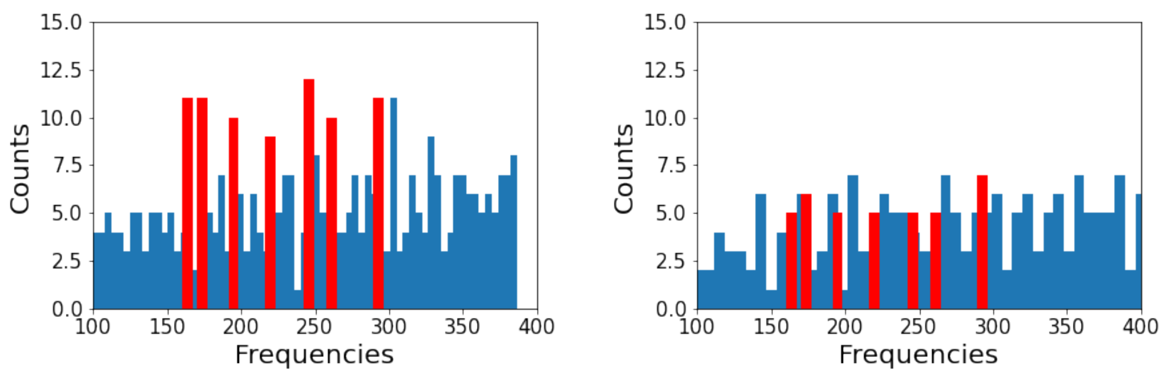
		Young's modulus	Poisson's ratio	Density
Turtle shell		**1.07 GPa**	**0.25**	**1.52 g/cm^3 1.47 g/cm^3 1.03 g/cm^3 (average 1.34 g/cm^3)**
Leather covering		**0.072 GPa**	**0.6 (appr. as 0.4999)**	**8.6 g/cm^3**

Figure 11: Material properties of the chelys.

To summarize, this program takes a set of dimensions of the chelys and calculates, given its material properties, what frequencies will resonate naturally (its eigenfrequencies). This information, combined with some knowledge of the chelys's music, can lead us to discover the optimal size(s) of the instrument—the size(s) that the Greeks would have chosen in order to produce the best-quality instrument.

Significant Results and Innovation

Having developed this program, I have tested different sizes of the chelys. While data analysis is ongoing, current significant results include evidence that larger models would have produced a greater number of lower eigenfrequencies. The histograms in Figure 12 demonstrate that the larger instrument (Figure 12a) produces a greater number of lower frequencies. Since the targeted pitches that the chelys was tuned to are also low frequencies, the larger instrument will play these notes better than the smaller instrument.



(a)

length: 30 cm

width: 22.5 cm

height: 6.75 cm

(b)

length: 20 cm

width: 15 cm

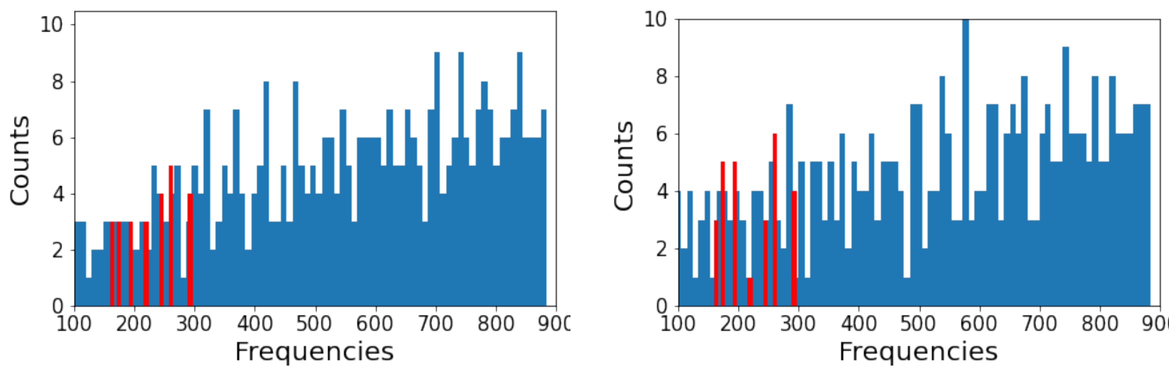
height: 5.6 cm

Figure 12: Comparing size to eigenfrequencies;

all eigenfrequencies within the chosen range are shown by blue bins;

red bins indicate eigenfrequencies near the targeted notes.

In addition, a comparison of how changing tortoise shell to wood is being made, with results showing that wood may resonate at the targeted frequencies better.



(a)

Tortoise Shell

(b)

American Red Oak

Figure 13: Comparing wood to tortoise shell, where size and shape are constant.

This program is unique in that it provides useful reconstructions when there is insufficient archaeological evidence to reconstruct an instrument from its remains. In addition, as the first major study on the chelys since the 2011 and 2015 studies [3][4], this project takes a different approach that can be used to learn more about how changing the chelys's size would have changed its eigenfrequencies.

Conclusion and Future Work

Results from this study show that my algorithm successfully generates data that help us better understand instruments such as the chelys. This project can be extended in many aspects, and refinements of the model may lead to higher accuracy. For example, I will improve the precision of the material parameters, such as the leather Poisson's ratio (which is currently a rough estimate) and account for the multiple layers of shell (which each have a different density). By taking a digital model approach, I have ensured that my program is usable for other instruments with other materials, such as closed drums made from wood or leather. Future studies may implement this tool to research other such ancient instruments. Finally, data is being collected for the building of the optimal chelys for players today, made out of widely available woods in the U.S. This will help increase public awareness of this instrument and the musical culture of the ancient Greeks.

Acknowledgments

I would like to acknowledge my project mentor, Paul Welch, for guiding me and giving me a deeper understanding of the science, computer science, and mathematics needed to put this project together. In addition, I would also like to thank the judges and reviewers associated with the Supercomputing Challenge and the Science Fair, who generously provided me feedback and encouragement throughout the year. Finally, I am grateful towards the authors of the numerous papers that contributed to my knowledge of the project's background and gave me further motivation for completing it.

Note

Figures 4-7 and all histograms were generated using matplotlib [18].

Appendix A

Pseudo-code for generating the tetrahedral mesh of the sound box is provided below.

```
#Set input parameters

#Calculate structural limits
xmin = c_x-r_x
xmax = c_x+r_x
zmin = c_z-r_z
zmax = c_z+r_z

#generate evenly-spaced points along major axis
Xraster = np.linspace(xmin, xmax, N_x, dtype=float)

#generate evenly-spaced points parallel to minor axis
Zraster = np.linspace(zmin, zmax, N_z, dtype=float)

#calculate point heights for top surface
Yraster = Get Heights(h_x, c_x, c_z, r_x, r_z, Xraster, Zraster,

#generates mesh using gmsh
MkModel3(Xraster, Yraster, Zraster, lc, N_x, N_z)
```

(a) Overview: Constructing The Mesh

```
Get Heights (h_x,c_x,c_z,r_x,r_z,Xraster,Zraster, N_x, N_z):

  y = numpy ([N_x,N_z]) matrix
  i_x = 0

  for x in Xraster:
    i_z = 0

    #Get vertex height
    h_z = -(h_x/r_x^2) (x-c_x)^2 + h_x

    #Get curvature parameter
    b_z = -h_z/[(r_z^2) ( 1 – ((x-c_x)/r_x)^2 ) ] || 0 if x on extremum

    for z in Zraster:
      if z at x is within the bounds of the ellipse:
        y[i_x][i_z] =  b_z (z-c_z)^2 + h_z || -1 if z outside
      i_z = i_z+1

    i_x = i_x+1
  return y
```

(b) Calculating Vertical Coordinates

```
FindBoundaries(Yraster, N_x, N_z):
  numbers = np.zeros(N_x, dtype=int)
  zindex0 = np.zeros(N_x, dtype=int)
  zindex1 = np.zeros(N_x, dtype=int)
  for i_x:
    counter1 = 0
    firstindex = -1
    lastindex = -1
    for i_z:
      if y at (i_x, i_z) is not flagged:
        counter1 += 1
        if firstindex=-1:
          firstindex = i_z
      if lastindex < 0 and firstindex > 0 and Yraster[i_x][i_z] < 0:
        lastindex =  i_z-1
    numbers[i_x] = counter1
    zindex0[i_x] = firstindex
    if lastindex < 0:
      lastindex = N_z - 1
    zindex1[i_x] = lastindex

  return numbers, zindex0, zindex1
```

Figure 15: Determining Boundaries

```
MkModel3(Xraster, Yraster, Zraster, lc, Nx, Nz)
    Points = [ ]
    PointMap = numpy ([Nx,Nz]) matrix

    #k cycles through points on each parabola and resets when the x value changes
    k = 0
    for each parabola along x:
            if zindex0 > -1 and numbers > 1 for this value of x:
            for iz from zindex0 to zindex1+1 for this value of x:
                Y = Yraster[ix][iz] || 0 if iz == zindex0 or iz == zindex1 for this value of x
                    thisPoint = gmsh.model.geo.add_point(Xraster[ix], Y, Zraster[iz], lc)
                    add thisPoint to list Points
                    PointMap[ix][iz] = k
                    k = k+1


    #Make Triangles for shell
    for each parabola along x:
        for each z-value on a parabola:
            connect points in sets of three with lines
            add curve loops
            mesh surfaces
```

Figure 16: Generating Mesh

Appendix B

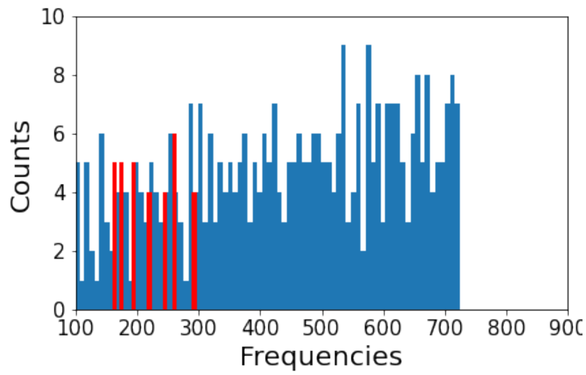
Current output from data runs is shown below, showing frequency (Hz) vs. number of frequencies. Blue bins indicate all eigenfrequencies within the specified range, whereas red bins indicate eigenfrequencies close to the targeted frequencies at which it was played. The taller the red bins, the better quality the instrument. Because my current leather Poisson's ratio is an estimate, sensitivity tests were conducted to test how big a difference this makes in the data.

Sensitivity Tests for Leather Poisson's Ratio

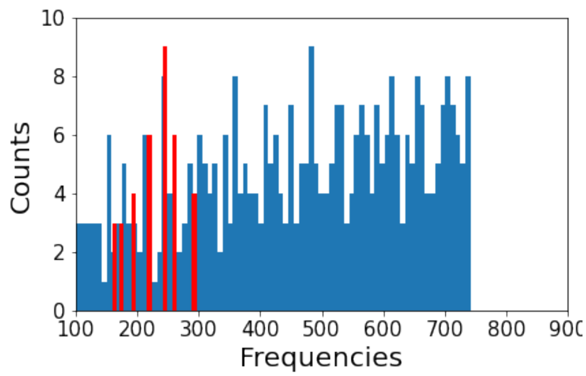
400 frequencies
materials: tortoise shell and leather
length: 15 cm
width: 11.2
thickness: 0.2

Leather Poisson's ratio

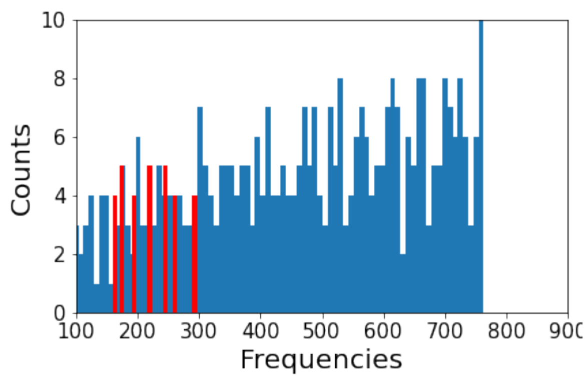
0.4



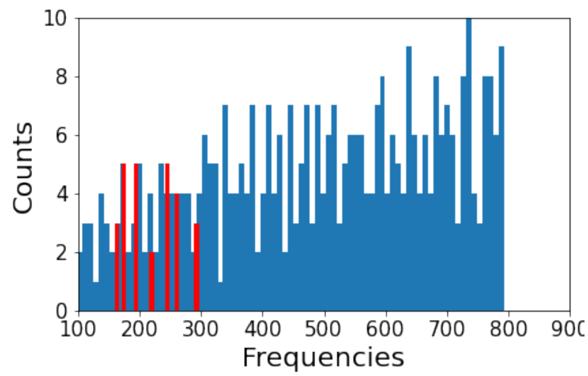
0.42



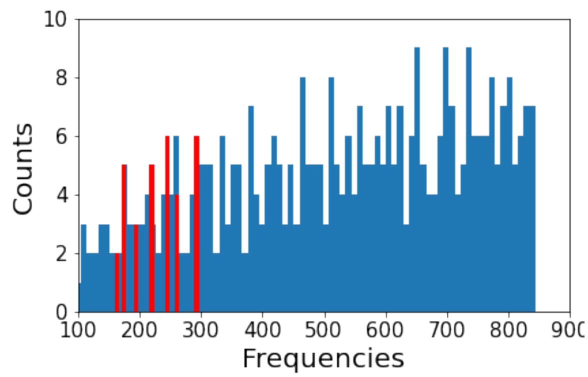
0.44



0.46


Counts
Frequencies
0
2
4
6
8
10
100
200
300
400
500
600
700
800
900

0.48


Counts
Frequencies
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2
4
6
8
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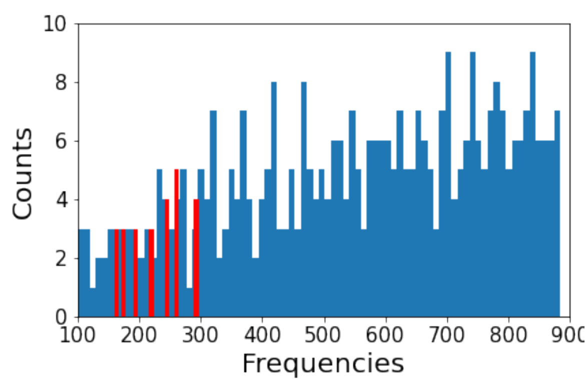
100
200
300
400
500
600
700
800
900

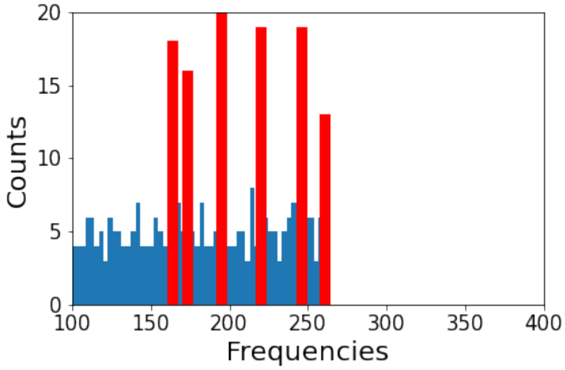
0.49


Counts
Frequencies
0
2
4
6
8
10
100
200
300
400
500
600
700
800
900

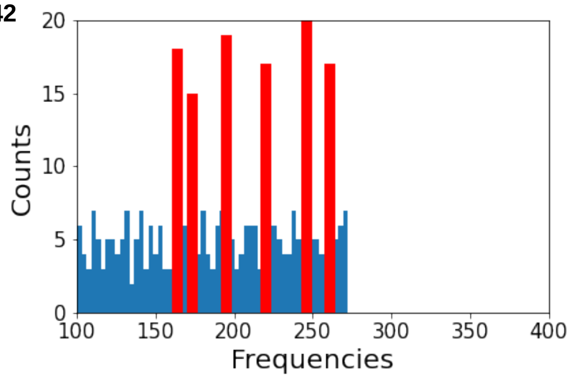
400 frequencies
materials: tortoise shell and leather
length: 30 cm
width: 22.5
thickness: 0.2

Leather Poisson's ratio

0.4



0.42



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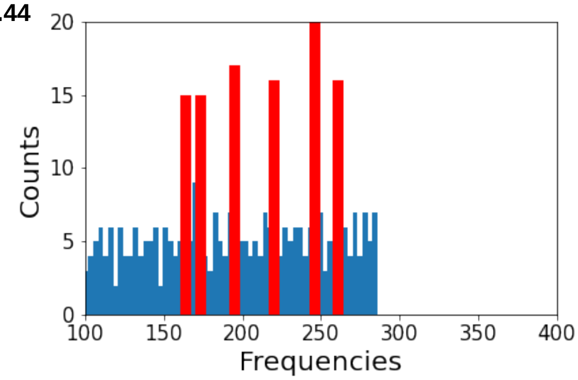

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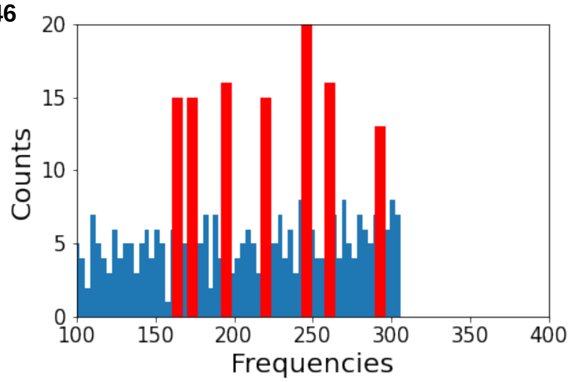
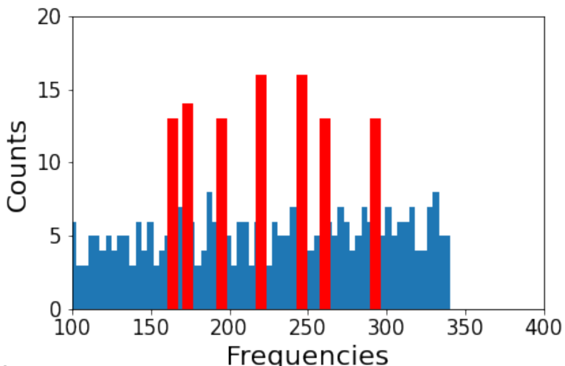


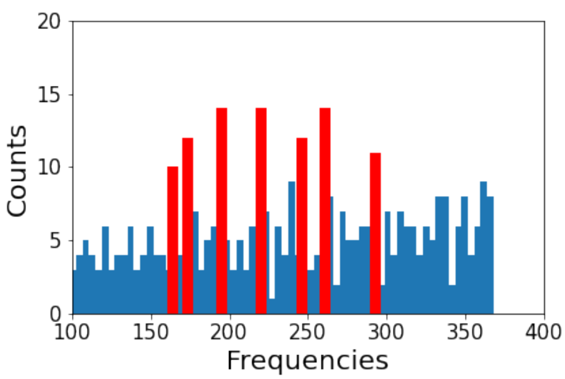
Counts
Frequencies

0.48



Counts
Frequencies

0.49



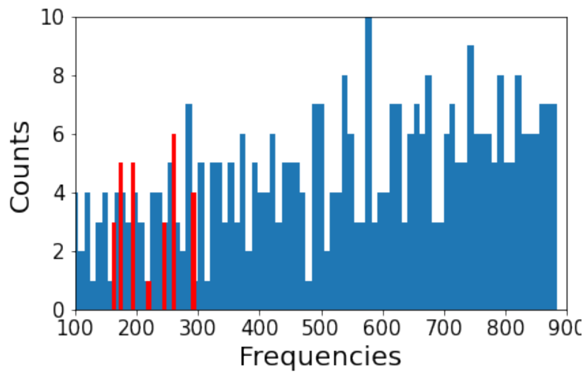
Counts
Frequencies

Comparing Eigenfrequencies of Wood to Tortoise Shell

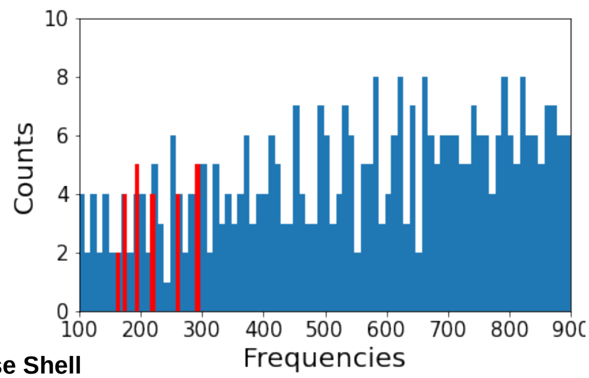
400 frequencies
materials: leather
length: 15 cm
width: 11.2
thickness: 0.2

Type of Material

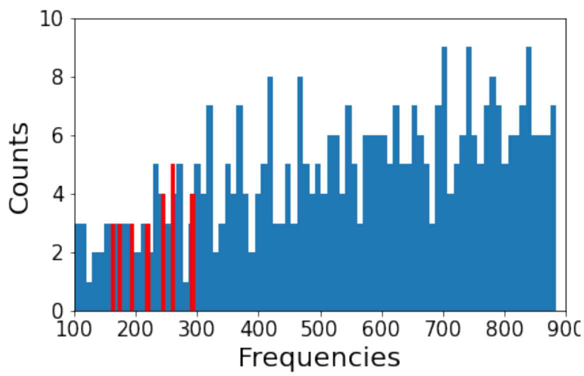
American Red Oak Wood



American White Oak Wood



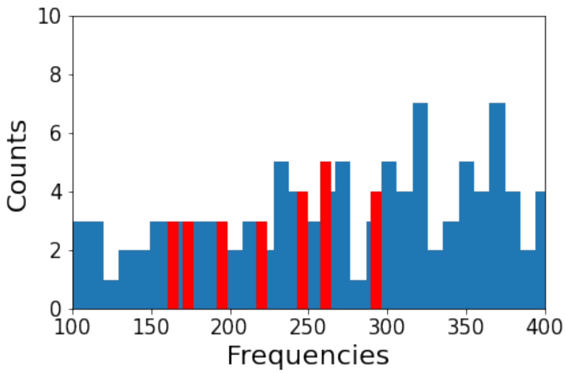
Tortoise Shell



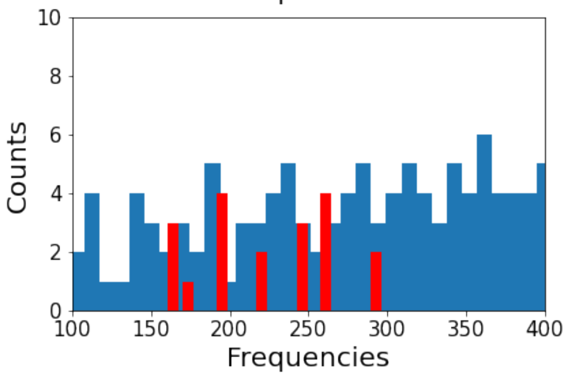
Comparing Sound Box Sizes

400 frequencies
materials: leather and tortoise shell
thickness: 0.2

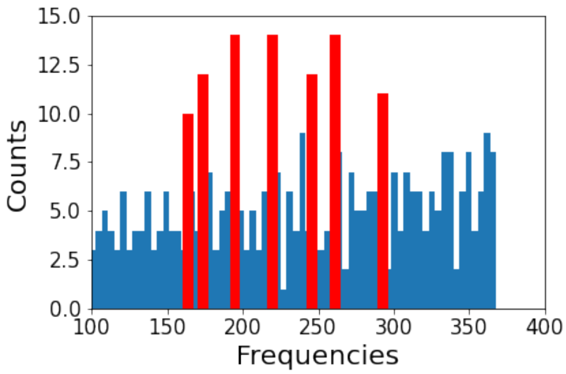
length: 15 cm
width: 11.2 cm
height: 5.6 cm



length: 15 cm
width: 11.2 cm
height: 6.75 cm



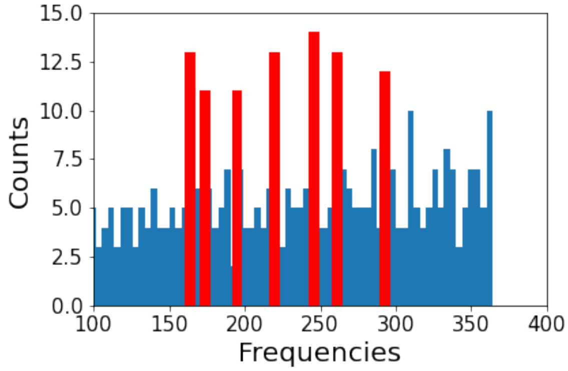
length: 30 cm
width: 22.5 cm
height: 5.6 cm



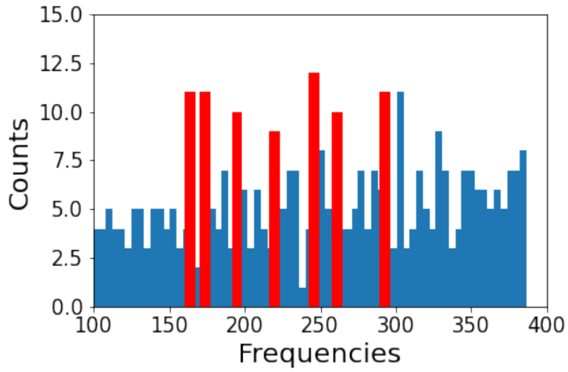
Comparing Sound Box Sizes

400 frequencies
materials: leather and tortoise shell
thickness: 0.2

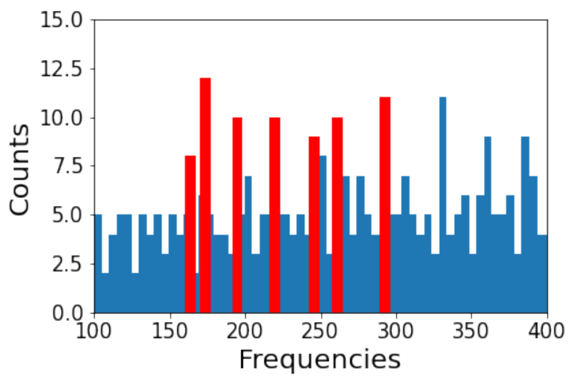
length: 30 cm
width: 22.5 cm
height: 6.75 cm



length: 29 cm
width: 21.75 cm
height: 5.6 cm

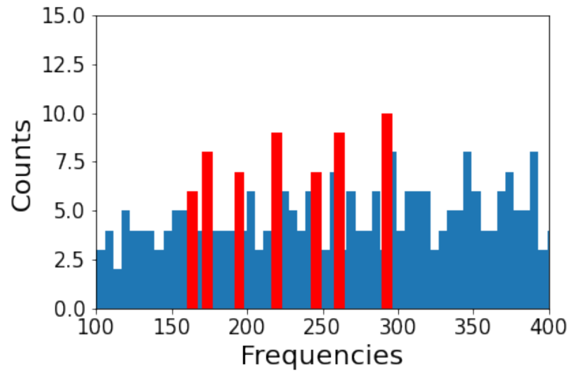


length: 26 cm
width: 19.5 cm
height: 5.6 cm

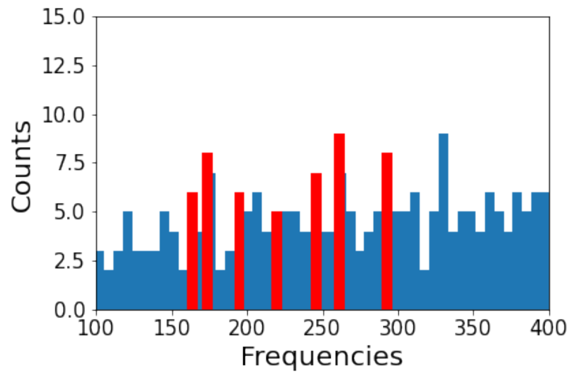


400 frequencies
materials: leather and tortoise shell
thickness: 0.2

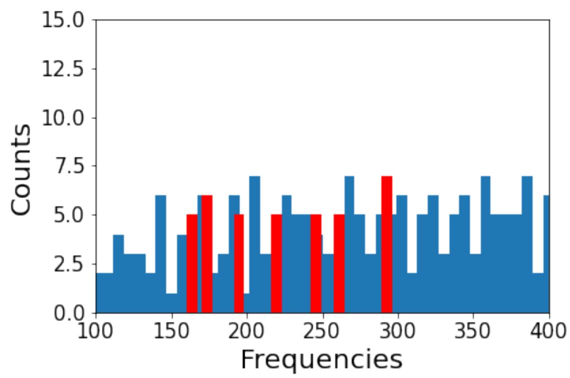
length: 24 cm
width: 18 cm
height: 5.6 cm



Length: 22 cm
width: 16.5 cm
height: 5.6 cm



length: 20 cm
width: 15 cm
height: 5.6 cm



References

[1] Eichmann, R. (2018). Music Archaeology. In: Bader, R. (eds) Springer Handbook of Systematic Musicology. Springer Handbooks. Springer, Berlin, Heidelberg. `https://doi.org/10.1007/978-3-662-55004-5_51`.

[2] Roda, Antonio and De Poli, Giovanni and Canazza, Sergio and Sun, Zezhou and Whiting, Emily. (2021). 3D virtual reconstruction and sound simulation of old musical instruments. 32. 359-374. 10.19282/ac.32.1.2021.20.

[3] Bakarezos, Efthimios; Vathis, Vasilios; Brezas, Spyros; Orphanos, Yannis; Papadogiannis, Nektarios A. "Acoustics of the Chelys–An ancient Greek tortoise-shell lyre," *Applied Acoustics* **2012**, *73*, 478-483.

[4] Koumartzis, Nikolaos. "The lyre 2.0 project : re-inventing an ancient greek artifact and re-introducing it as a product the modern world." (2014). `http://hdl.handle.net/11544/375`. Accessed 01 April 2024.

[5] "Wooden Lyre with tortoise shell sound-box; restored from remains". *British Museum*, `https://www.britishmuseum.org/collection/image/253898001`, Accessed 01 March 2024.

[6] Yokoyama, Masao; Takei, Amane; Shioya, Ryuji; Yagawa, Genki. "Coupled simulation of vibration and sound field of Stradivari's violin," *Proceedings of Meetings on Acoustics* **2023**, *51*, 035001.

[7] Forinash, K. and Wolfgang Christian. *Sound: An Interactive eBook*. 2024. Accessed 30 March 2024.

[8] Romani Mistretta, Marco. "Hermes the Craftsman: The Invention of the Lyre". *GAIA. Revue interdisciplinaire sur la Grèce ancienne*, 20, 1, 2017, pp. 5-22, `doi:10.3406/gaia.2017.1716`.

[9] Eucharides Painter. Terracotta Amphora (jar). 07.286.78. CE 490 BCE, Metropolitan Museum of Art, New York.

[10] Geuzaine, C.; Remacle, J.-F. “Gmsh: a three-dimensional finite element mesh generator with built-in pre- and post-processing facilities,” *International Journal for Numerical Methods in Engineering* **2009**, *79*, 1309-1331.

[11] Dhondt, G. The Finite Element Method for Three-Dimensional Thermomechanical Applications, Wiley, 2004.

[12] Wei Zhang, Chengwei Wu, Chenzhao Zhang, Zhen Chen, Microstructure and mechanical property of turtle shell, Theoretical and Applied Mechanics Letters, Volume 2, Issue 1, 2012, 014009, ISSN 2095-0349, `https://doi.org/10.1063/2.1201409`.

[13] Mihai A, Seul A, Curteza A, Costea M. Mechanical Parameters of Leather in Relation to Technological Processing of the Footwear Uppers. Materials (Basel). 2022 Jul 22;15(15):5107. `doi:10.3390/ma15155107.PMID:35897538;PMCID:PMC9331295`.

[14] The Engineering ToolBox (2009). Solids - Densities. [online] Available at: `https://www.engineeringtoolbox.com/density-solids-d_1265.html`. [Accessed 17 February 2024].

[15] B. Alheit, S. Bargmann, B.D. Reddy, “Computationally modelling the mechanical behaviour of turtle shell sutures—A natural interlocking structure”, Journal of Mechanical Behavior of Biomedical Materials, vol. 110, 2020, 103973, ISSN 1751-6161, `https://doi.org/10.1016/j.jmbbm.2020.103973`.

[16] Pollock C, Kanis C. Basic information sheet: Marginated tortoise. March 18, 2015. LafeberVet Web site. Available at `https://lafeber.com/vet/basic-information-sheet-marginated-tortoise/`.

[17] “Tortoise Shell”, Scientific American Magazine, vol. 8, No. 46, July 1853, p. 368.

[18] Hunter, J. D. "Matplotlib: A 2D graphics environment", *Computing in Science & Engineering* 9, 3, https://doi.org/10.1109/MCSE.2007.55 (2007), 90-95.